C2 Exercise 2A (the sine rule)

Note Title 20/01/2007

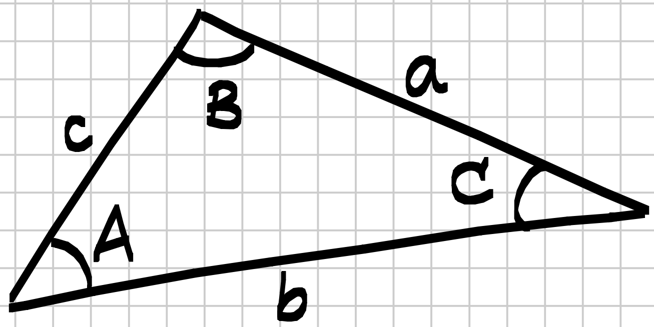
STATEMENT OF RULE

Throughout this exercise use

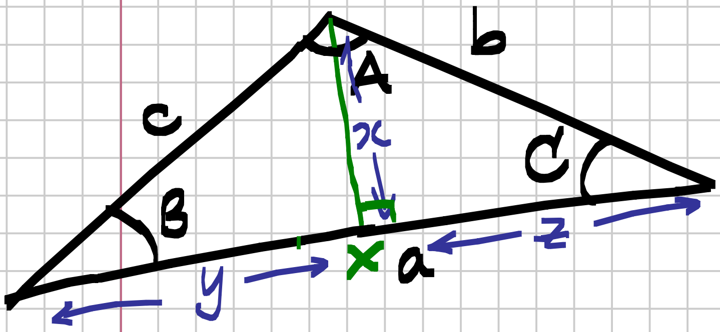
The sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or its reciprocal: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

In the triangle



Proof part 1



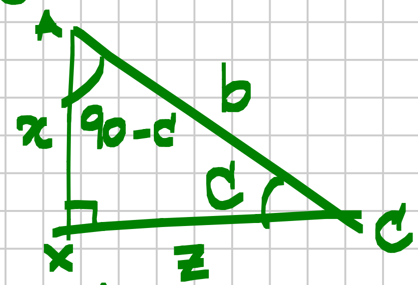
Assume WLOG that $A > B > C$

Add point X on BC such that AXC is a right angle.

Call AX length x.

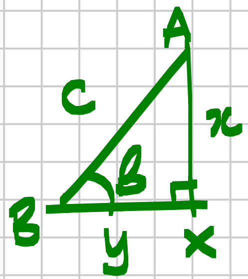
Call BX length y, and CX length z. Then $a = y + z$.

Consider $\triangle AXC$



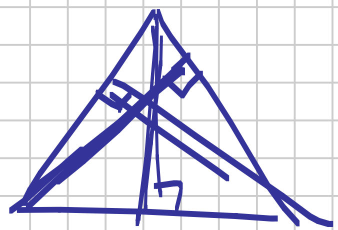
$x = b \sin C$

Consider $\triangle AXB$



$x = c \sin B$

$\Rightarrow \quad b \sin C = c \sin B$

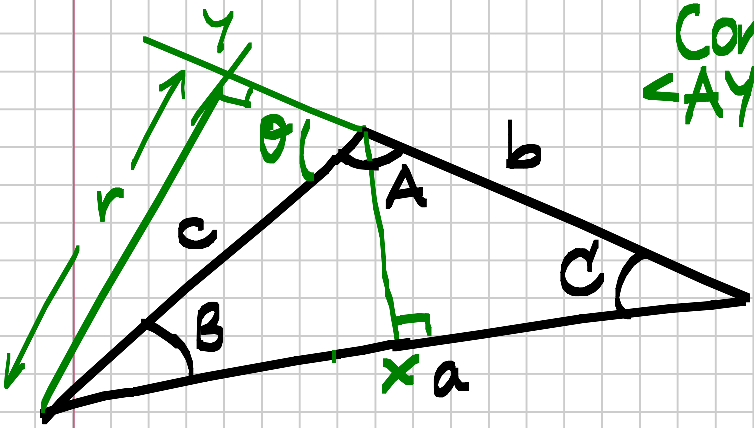


$$\Rightarrow \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (1)$$

PROOF part 2

If $A < 90$ re apply method above, else if $A > 90$:

Consider point Y on CA produced s.t. $\angle AYB = 90°$



$r = a \sin C$

$r = c \sin \theta$

Now since $\sin x = \sin(180 - x)$

we have $r = c \sin(180 - \theta)$

$\Rightarrow r = c \sin A$ because $\theta = 180 - A$

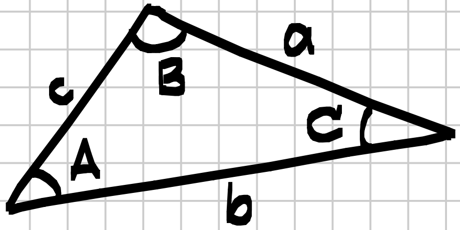
So $a \sin C = c \sin A$

and hence $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ ②

and combining ① & ② we get $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

C2 Ex 2A

1



It's good style to use a small line at the top of a capital C to distinguish it from a carelessly oversized lowercase c.

a) Given that $a = 8\text{cm}$, $A = 30°$, $B = 72°$, find b.

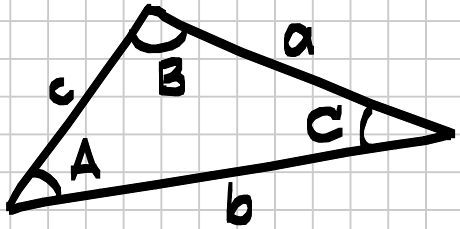
$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow b = \frac{a \sin B}{\sin A}$$

$$b = \frac{8 \sin 72°}{\sin 30°} = 15.216904260722\ldots$$

$b \simeq 15.2$ cm (to 3 sig. fig.)

C2 Ex 2A

1



b) Given that $a = 24\text{ cm}$, $A = 110°$, $C = 22°$, find c

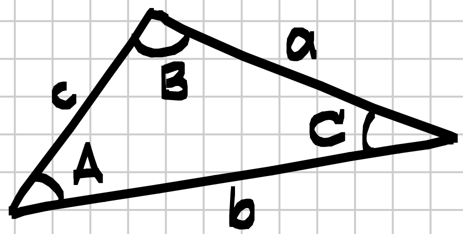
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow c = \frac{a \sin C}{\sin A}$$

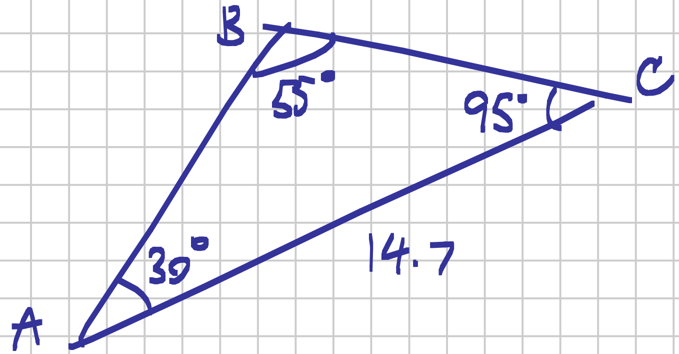
$$c = \frac{24 \sin 22°}{\sin 110°} = 9.567552243267\ldots$$

$$c \simeq 9.57\text{cm} \quad (\text{to } 3 \text{ sig. fig.})$$

C2 Ex 2A

1





c) Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find a.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a = \frac{b \sin A}{\sin B}$$

but we don't have B. Use $A + B + C = 180$

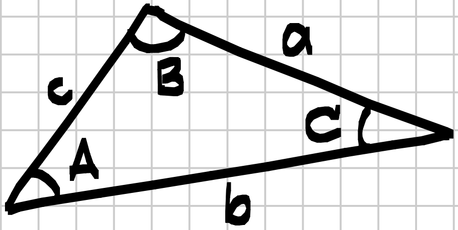
$\Rightarrow \quad B = 55°$

so $\quad a = \dfrac{14.7 \sin 30°}{\sin 55°} = 8.972693227396\ldots$

$a = 8.97$ cm (to 3 sig. fig.)

C2 Ex 2A

1



d) Given that $c = 9.8\text{cm}$, $B = 68.4°$, $C = 83.7°$, find a.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow a = \frac{c \sin A}{\sin C}$$

$$A = 180 - (68.4 + 83.7) = 27.9$$

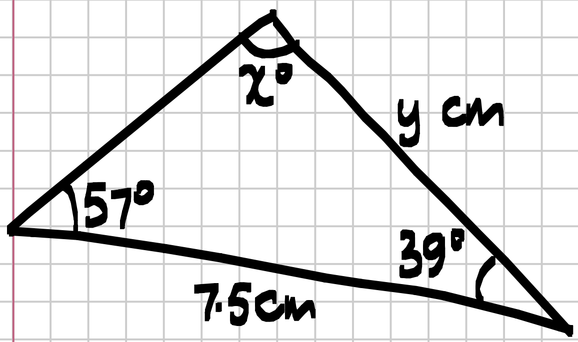
$$a = \frac{9.8 \sin(27.9)}{\sin(83.7)} = 4.613573757 1\ldots$$

$$a \approx 4.61 \text{ cm} \quad (\text{to } 3 \text{ sig. fig.})$$

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

a)



$$x = 180 - (57 + 39)$$

$$x = 84°$$

$$\frac{y}{\sin 57} = \frac{7.5}{\sin x}$$

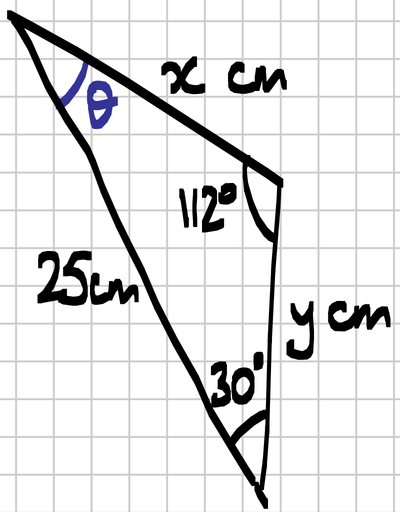
$$\Rightarrow \quad y = \frac{7.5 \sin 57°}{\sin 84°} = 6.324676499$$

$$y \simeq 6.32 \text{ cm}$$

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

b)



Add θ to diagram.

$$\theta = 180 - (112 + 30)$$

$$\theta = 38°$$

$$\frac{25}{\sin 112} = \frac{x}{\sin 30} = \frac{y}{\sin\theta}$$

$$x = \frac{25\sin 30°}{\sin 112°} = 13.48168428 \simeq 13.5 \text{ cm}$$

$$y = \frac{25\sin 38°}{\sin 112°} = 16.60030727 \simeq 16.6 \text{ cm}$$

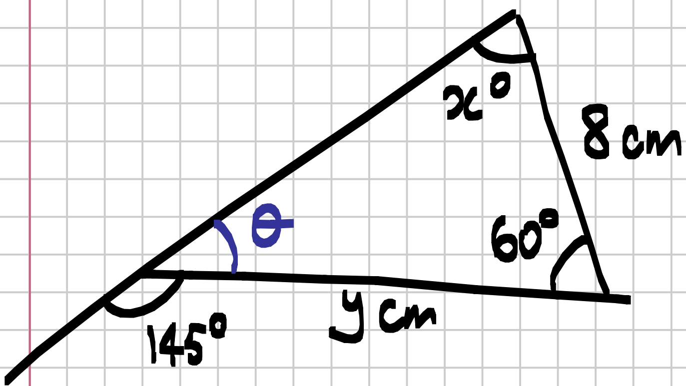
θ = "theta"

ϕ = "phi"

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

c)



Add θ to diagram

$\theta = 180 - 145 = 35°$

$x = 180 - (60 + \theta) = 180 - 95 = 85°$

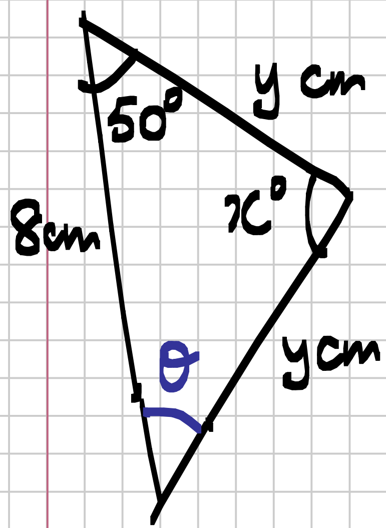
$$\frac{y}{\sin x} = \frac{8}{\sin \theta}$$

$$\Rightarrow \quad y = \frac{8 \sin 85°}{\sin 35°} = 13.89449963 \simeq 13.9 \text{ cm.}$$

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

d)



Add θ to diagram & note isosceles △

Hence $\theta = 50°$ so $x = 180 - (50+50) = 80°$

$$\frac{y}{\sin 50} = \frac{8}{\sin x}$$

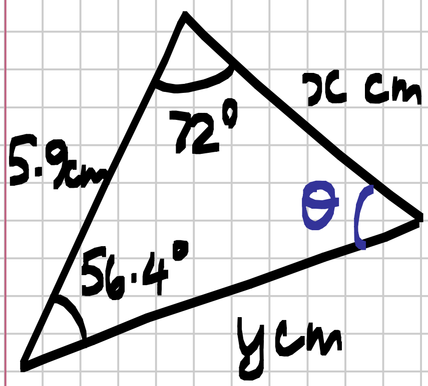
$$\Rightarrow \quad y = \frac{8 \sin 50}{\sin 80} = 6.222895307$$

$$y \simeq 6.22 \text{ cm}$$

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

e)



Add θ to diagram

$$\theta = 180 - (72 + 56.4) = 51.6°$$

$$\frac{5.9}{\sin\theta} = \frac{x}{\sin 56.4} = \frac{y}{\sin 72}$$

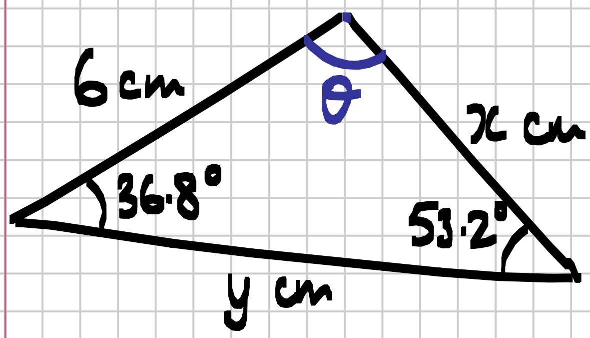
$$\Rightarrow \quad x = \frac{5.9 \sin 56.4}{\sin 51.6} = 6.270609094 \simeq 6.27 \text{ cm}$$

$$\Rightarrow \quad y = \frac{5.9 \sin 72}{\sin 51.6} = 7.159985060 \simeq 7.16 \text{ cm}$$

C2 Ex 2A

2 In each of the following triangles calculate the values of x and y.

f)



Add angle θ to the diagram

$\theta = 180 - (36 \cdot 8 + 53 \cdot 2)$

$\theta = 90°$

Stop! It's a right-angled triangle...

Use of sine rule unnecessary. Can use standard trig. relations.

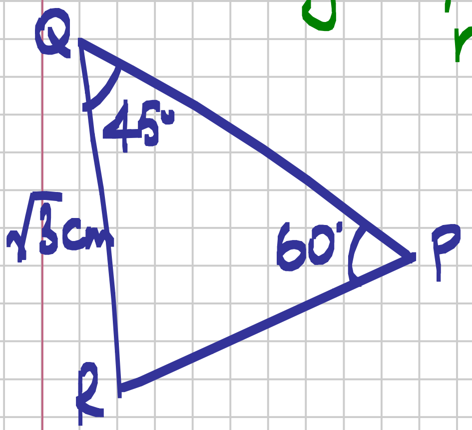
$\sin 53 \cdot 2 = \dfrac{6}{y} \Rightarrow y = \dfrac{6}{\sin 53 \cdot 2} = 7 \cdot 49314966 \simeq 7.49\text{cm}$

$\tan 53 \cdot 2 = \dfrac{6}{x} \Rightarrow x = \dfrac{6}{\tan 53 \cdot 2} = 4.488573474 \simeq 4.49\text{ cm}$

C2 Ex 2A

3 In $\triangle PQR$, $QR = \sqrt{3}$ cm, $\angle PQR = 45°$ and $\angle QPR = 60°$.
Find PR and PQ.

Draw a diagram, try to make the angles roughly accurate, but remember it's only a sketch.



$$\frac{\sqrt{3}}{\sin 60} = \frac{PR}{\sin 45} = \frac{PQ}{\sin(P\hat{R}Q)}$$

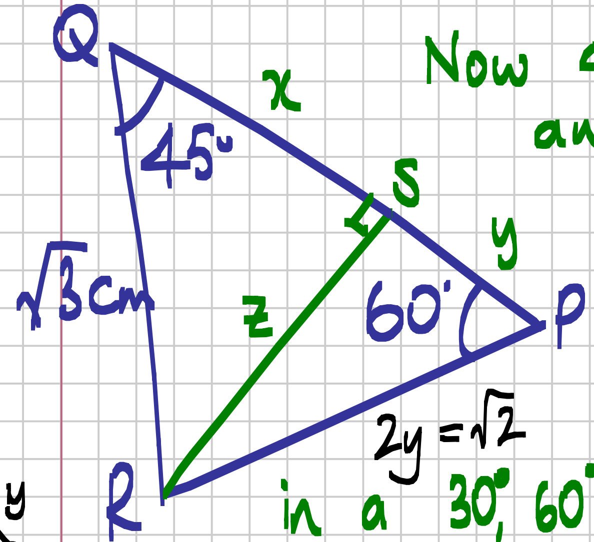
$$P\hat{R}Q = 180 - (60 + 45) = 75°$$

$$PR = \frac{\sqrt{3}\ \sin 45}{\sin 60} = 1.414213562 \simeq 1.41 \text{ cm}$$

$$PQ = \frac{\sqrt{3}\ \sin 75°}{\sin 60} = 1.931851653 \simeq 1.93 \text{ cm}$$

C2 Ex 2A

3 Revisited in <u>surds</u>.



Add point S on PQ such that $P\hat{S}R = 90°$

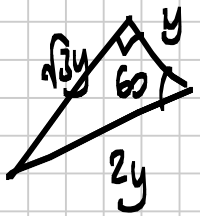
Call $QS = x$, $SP = y$ and $SR = z$

Now $\triangle QSR$ is an isosceles right triangle $\Rightarrow x = z$

and $x^2 + z^2 = (\sqrt{3})^2$ by Pythagoras' Theorem

$\Rightarrow 2x^2 = 3 \quad \Rightarrow x^2 = 3/2$

$\Rightarrow x = z = \sqrt{3/2}$ or rationalising $x = z = \dfrac{\sqrt{6}}{2}$



in a 30°, 60°, 90° triangle the adjacent is half the hypotenuse

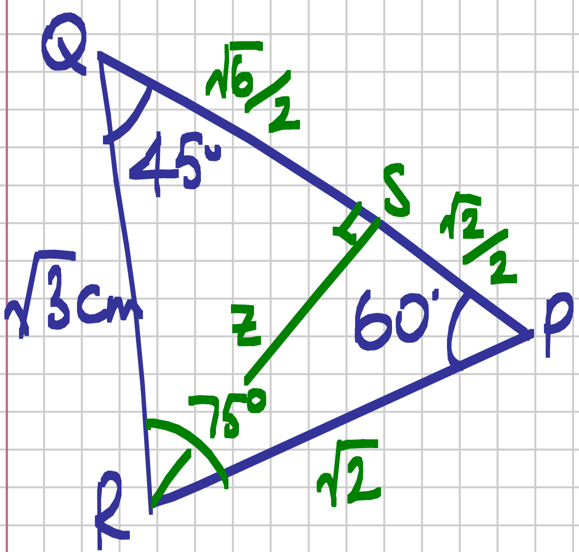
so $(2y)^2 = y^2 + z^2 \quad \Rightarrow \quad 4y^2 - y^2 = \dfrac{3}{2}$

$\Rightarrow \quad 3y^2 = \dfrac{3}{2} \quad \Rightarrow \quad y^2 = \dfrac{1}{2} \quad \Rightarrow \quad y = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad y = \dfrac{\sqrt{2}}{2}$

Hence $PR = 2y = \sqrt{2}$ and $PQ = x + y = \dfrac{\sqrt{6} + \sqrt{2}}{2} = \dfrac{\sqrt{2}(\sqrt{3}+1)}{2}$

C2 Ex2A

3 Extended.



Having established these lengths as surds we can now evaluate sin 75° exactly.

$$\frac{\sin 75}{QP} = \frac{\sin 45}{\sqrt{2}} \quad \text{by sine Rule}$$

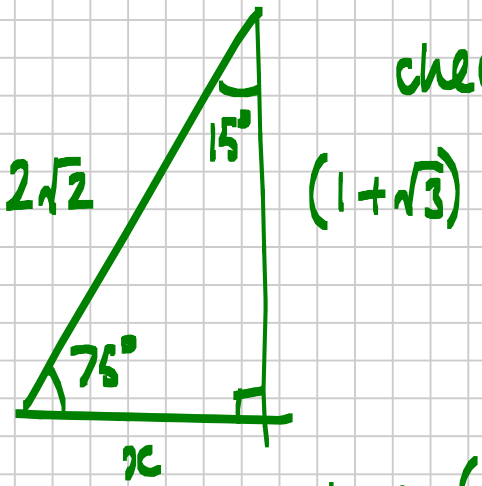
$$\Rightarrow \sin 75 = \frac{QP \sin 45}{\sqrt{2}}$$

Now $QP = \dfrac{\sqrt{2}(1+\sqrt{3})}{2}$ ✓ and $\sin 45 = \dfrac{\sqrt{2}}{2}$ from the standard result. ✓

$$\Rightarrow \sin 75 = \frac{\sqrt{2}(1+\sqrt{3}) \times \cancel{\sqrt{2}}}{2 \times 2 \times \cancel{\sqrt{2}}} = \frac{\sqrt{2}(1+\sqrt{3})}{4} \checkmark \text{ or } \frac{\sqrt{2}+\sqrt{6}}{4}$$

C2 Ex 2A

3 ext. We can take this result & derive $\sin 15°$, $\cos 75$, $\cos 15$, $\tan 15$ and $\tan 75$ in the triangle:



check that in this Δ $\sin 75 = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$

Now $x^2 = (2\sqrt{2})^2 - (1+\sqrt{3})^2$

$\Rightarrow x^2 = 4\times 2 - (1 + 2\sqrt{3} + 3)$

$\Rightarrow x^2 = 4 - 2\sqrt{3}$

but $(\sqrt{3}-1)^2 = 3 - 2\sqrt{3} + 1 = 4 - 2\sqrt{3}$ so $x = \sqrt{3} - 1$

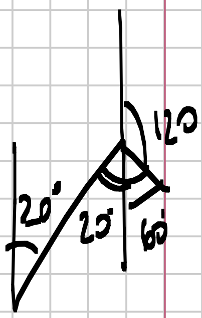
$\tan 15° = \frac{\sqrt{3}-1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{2\sqrt{3}-4}{-2} = 2-\sqrt{3}$ ✓, $\cos 75 = \sin 15° = \frac{x}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ ✓

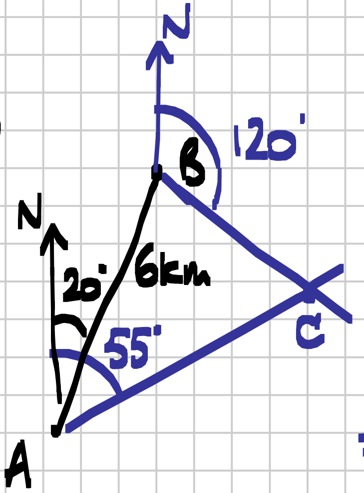
$\sin 75° = \cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$ ✓; $\tan 75 = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = 2+\sqrt{3}$ ✓

what other angles can you work out exactly as surds?

C2 Ex2A

4 Town B is 6km on a bearing of 020° from town A. Town C is located on a bearing of 055° from town A and on a bearing of 120° from town B. Work out the distance of town C from town A and town B.

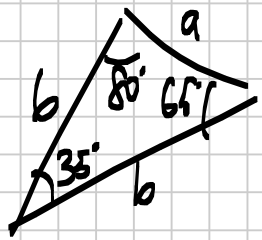




$\angle BAC = 55 - 20 = 35$

$\angle ABC = 20° + (180 - 120) = 80°$.

$\angle ACB = 180 - (80 + 35) = 65$



$$\frac{BC}{\sin B\hat{A}C} = \frac{AB}{\sin A\hat{C}B}$$

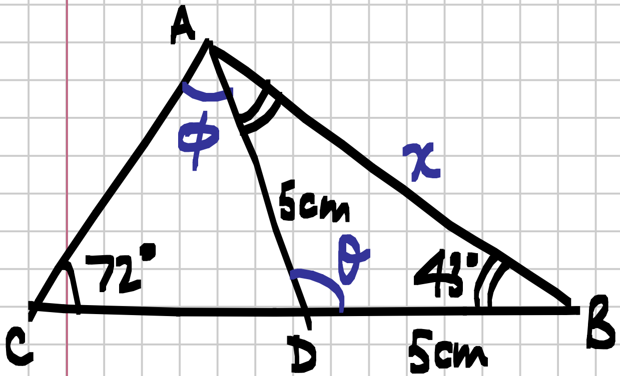
$\Rightarrow BC = 6 \sin 35 / \sin 65 = 3.797229448 \simeq 3.80$ km

similarly $AC = \dfrac{6 \sin 80°}{\sin 65°} = 6.519690775 \simeq 6.52$km

so C is 6.52 km from A and 3.80km from B.

C2 Ex2A

5 In the diagram AD = DB = 5cm, $\angle ABC = 43°$ and $\angle ACB = 72°$
Calculate AB and CD.



Consider $\triangle ABD$. Mark $\theta = A\hat{D}B$ and $x = AB$

$\theta = 180 - 2 \times 43 = 94°$

$$\frac{x}{\sin\theta} = \frac{5}{\sin 43} \Rightarrow x = \frac{5\sin 94}{\sin 43}$$

So $AB = x = 7.313537016 \simeq 7.31\text{cm}$.

Now $\phi = \theta - 72 = 22°$ by external angle of $\triangle$ rule

$$\frac{CD}{\sin\phi} = \frac{5}{\sin 72} \Rightarrow CD = \frac{5\sin 22}{\sin 72} = 1.969423411 \simeq 1.97\text{cm}$$